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Journal of Geometry and Physics 55 (2005) 50–74
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

New strings for old Veneziano amplitudes I. Analytical treatment

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Received 22 April 2004; received in revised form 22 October 2004; accepted 27 November 2004

Available online 25 January 2005

Abstract

The bosonic string theory evolved as an attempt to find a physical/quantum mechanical model capable of reproducing Euler's beta function (Veneziano amplitude) and its multidimensional analogue. The multidimensional analogue of beta function was studied mathematically for some time from different angles by mathematicians such as Selberg, Weil and Deligne, among many others. The results of their studies apparently were not taken into account in physics literature on string theory. In a recent publication [IJMPA 19 (2004) 1655] an attempt was made to restore the missing links. The results of this publication are incomplete, however, since no attempts were made at reproduction of known spectra of both open an closed bosonic strings or at restoration of the underlying model(s) reproducing such spectra. Nevertheless, as discussed in this publication the existing mathematical interpretation of the multidimensional analogue of Euler's beta function as one of the periods associated with the corresponding differential form "living" on the Fermat-type (hyper)surfaces, happens to be crucial for restoration of the quantum/statistical mechanical model reproducing such generalized beta function. Unlike the traditional formulations, this model is supersymmetric. Details leading to restoration of this model will be presented in the forthcoming Parts 2–4 of our work. They are devoted, respectively, to the group-theoretic, symplectic and combinatorial treatments of this model. In this paper the discussion is restricted mainly to the study of analytical properties of the multiparticle Veneziano and Veneziano-like (tachyon-free) amplitudes. In the last case, we demonstrate that the Veneziano-like amplitudes alone (with parameters adjusted accordingly) are capable of reproducing known spectra of both open and closed bosonic strings. The choice of parameters is subject to some

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doi:10.1016/j.geomphys.2004.11.008

constraints dictated by the mathematical interpretation of these amplitudes as periods of Fermat-type (hyper)surfaces considered as complex manifolds of Hodge-type.

PACS: 11.25.−w; 02.10.Rn; 02.10.Eb; 02.20.Df; 02.20.Tw

MSC: 83T30; 47A40

Subj. Class.: String theory; Scattering amplitudes

Keywords: Veneziano and the Veneziano-like amplitudes; Multidimensional beta and hypergeometric functions; Fermat hypersurfaces; Period integrals for Hodge-type manifolds

1. Brief review of the Veneziano amplitudes

In 1968 Veneziano [1] postulated the 4-particle scattering amplitude $A(s, t, u)$ given (up to a common constant factor) by

$$A(s, t, u) = V(s, t) + V(s, u) + V(t, u), \tag{1.1}$$

where

$$V(s, t) = \int_0^1 x^{-\alpha(s)-1}(1 - x)^{-\alpha(t)-1}\, \mathrm{d}x \equiv B(-\alpha(s), -\alpha(t)) \tag{1.2}$$

is Euler's beta function and $\alpha(x)$ is the Regge trajectory usually written as $\alpha(x) = \alpha(0) + \alpha' x$ with $\alpha(0)$ and α' being the Regge slope and the intercept, respectively. In the case of space–time metric with signature $\{-, +, +, +\}$, the Mandelstam variables s, t and u entering the Regge trajectory are defined by [2]

$$s = -(p_1 + p_2)^2, \qquad t = -(p_2 + p_3)^2, \qquad u = -(p_3 + p_1)^2. \tag{1.3}$$

The 4-momenta p_i are constrained by the energy–momentum conservation law leading to relation between the Mandelstam variables:

$$s + t + u = \sum_{i=1}^{4} m_i^2. \tag{1.4}$$

Veneziano [1] noticed[1] that to fit experimental data the Regge trajectories should obey the constraint

$$\alpha(s) + \alpha(t) + \alpha(u) = -1 \tag{1.5}$$

consistent with Eq. (1.4) in view of definition of $\alpha(s)$.

Remark 1.1. The Veneziano condition, Eq. (1.5), can be rewritten in a more general form. Indeed, let m, n, l be some integers such that $\alpha(s)m + \alpha(t)n + \alpha(u)l = 0$, then by adding this

[1] To get our Eq. (1.5) from Eq. (7) of Veneziano paper, it is sufficient to notice that his $1 - \alpha(s)$ corresponds to ours $-\alpha(s)$.

equation to Eq. (1.5) we obtain, $\alpha(s)\tilde{m} + \alpha(t)\tilde{n} + \alpha(u)\tilde{l} = -1$, or, more generally, $\alpha(s)\tilde{m} + \alpha(t)\tilde{n} + \alpha(u)\tilde{l} + \tilde{k} \cdot 1 = 0$. Both equations have been studied extensively in the book by Stanley [3] and play a major role in developments to be presented in this work and in Parts 2–4, which follow.

Veneziano noticed that with the help of the constraint, Eq. (1.5), the amplitude $A(s, t, u)$ can be equivalently written as follows:

$$A(s,t,u) = \Gamma(-\alpha(s))\Gamma(-\alpha(t))\Gamma(-\alpha(u))[\sin\pi(-\alpha(s)) + \sin\pi(-\alpha(t)) + \sin\pi(-\alpha(u))]. \tag{1.6}$$

The Veneziano amplitude looks strikingly similar to that suggested a bit later by Virasoro [4]. The latter (up to a constant) is given by

$$\bar{A}(s,t,u) = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b)\Gamma(b+c)\Gamma(c+a)} \tag{1.7}$$

with parameters $a = -\frac{1}{2}\alpha(s)$, etc. also subjected to the constraint:

$$\tfrac{1}{2}(\alpha(s) + \alpha(t) + \alpha(u)) = -1. \tag{1.8}$$

Use of the formulas

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin\pi x} \tag{1.9a}$$

and

$$4\sin x \sin y \sin z = \sin(x+y-z) + \sin(y+z-x) + \sin(z+x-y) - \sin(x+y+z) \tag{1.9b}$$

permits us to rewrite Eq. (1.7) in the alternative form (up to an unimportant constant):

$$\bar{A}(s,t,u) = [\Gamma(-\tfrac{1}{2}\alpha(s))\Gamma(-\tfrac{1}{2}\alpha(t))\Gamma(-\tfrac{1}{2}\alpha(u))]^2[\sin\pi(-\tfrac{1}{2}\alpha(s)) + \sin\pi(-\tfrac{1}{2}\alpha(t)) + \sin\pi(-\tfrac{1}{2}\alpha(u))]. \tag{1.10}$$

Although these two amplitudes look deceptively similar, mathematically, they are markedly different. Indeed, by using Eq. (1.6) conveniently rewritten as

$$A(a,b,c) = \Gamma(a)\Gamma(b)\Gamma(c)[\sin\pi a + \sin\pi b + \sin\pi c] \tag{1.11}$$

and exploiting the identity

$$\cos\frac{\pi z}{2} = \frac{\pi^z}{2^{1-z}}\frac{1}{\Gamma(z)}\frac{\zeta(1-z)}{\zeta(z)},$$

after some trigonometric calculations the following result is obtained:

$$A(a,b,c) = \frac{\zeta(1-a)}{\zeta(a)}\frac{\zeta(1-b)}{\zeta(b)}\frac{\zeta(1-c)}{\zeta(c)}, \tag{1.12}$$

provided that

$$a + b + c = 1. \tag{1.13}$$

For the Virasoro amplitude, apparently, no result like Eq. (1.12) can be obtained. As the rest of this paper demonstrates, the differences between the Veneziano and the Virasoro amplitudes are much more profound. The result, Eq. (1.12), is also remarkable in the sense that it allows us to interpret the Veneziano amplitude from the point of view of number theory, the theory of dynamical systems, etc. Details can be found in our recent work [5]. No such interpretation is possible to our knowledge for the Virasoro amplitudes. For this and other reasons to be described below in this paper, we shall consider only the Veneziano and Veneziano-like amplitudes.

In particular, now we would like to discuss some basic analytic properties of the 4-particle Veneziano amplitude. To this purpose we need to use the following identities:

$$\sin \pi z = \pi z \prod_{k=1}^{\infty} \left(1 - \left(\frac{z}{k}\right)\right) \left(1 + \left(\frac{z}{k}\right)\right) \tag{1.14}$$

and

$$\frac{1}{\Gamma(z)} = z\, \mathrm{e}^{-Cz} \prod_{k=1}^{\infty} \left(1 + \left(\frac{z}{k}\right)\right) \mathrm{e}^{-z/k} \tag{1.15}$$

with C being the Euler's constant

$$C = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right).$$

When combined with the Veneziano condition, $\alpha(s) + \alpha(t) + \alpha(u) = -1$, Eq. (1.5), the above results allow us to write (up to a constant factor) a typical singular portion of the Veneziano amplitude (the tachyons are to be considered separately):

$$\begin{aligned} A(s,t,u) = {} & \frac{1}{nlm\left(1 - \frac{\alpha(s)}{n}\right)} \frac{1}{\left(1 - \frac{\alpha(t)}{m}\right)} \frac{1}{\left(1 - \frac{\alpha(u)}{l}\right)} \\ & \times \left[\left(1 - \frac{\alpha(s)}{n}\right) C(n) + \left(1 - \frac{\alpha(t)}{m}\right) C(m) + \left(1 - \frac{\alpha(u)}{l}\right) C(l)\right], \end{aligned} \tag{1.16}$$

where $C(n)$, etc. are known constants and m, n, l are some non-negative integers. For actual use of this result the explicit form of these constants may be important. Looking at Eq. (1.14) we obtain,

$$C(n, \alpha) = \pi\alpha \frac{1}{\left(1 - \frac{\alpha}{n}\right)} \prod_{k=1}^{\infty} \left(1 - \left(\frac{\alpha}{k}\right)\right) \left(1 + \left(\frac{\alpha}{k}\right)\right), \tag{1.17}$$

where α can be $\alpha(s)$, etc. Clearly, this definition leads to further simplifications, e.g. to the manifestly symmetric form:

$$A(s,t,u)=\frac{1}{nlm}\left[\frac{C(n,\alpha(s))}{\left(1-\frac{\alpha(t)}{m}\right)}\frac{1}{\left(1-\frac{\alpha(u)}{l}\right)}+\frac{C(m,\alpha(t))}{\left(1-\frac{\alpha(s)}{n}\right)}\frac{1}{\left(1-\frac{\alpha(u)}{l}\right)}\right.$$
$$\left.+\frac{C(l,\alpha(u))}{\left(1-\frac{\alpha(s)}{n}\right)}\frac{1}{\left(1-\frac{\alpha(t)}{m}\right)}\right]. \tag{1.18a}$$

Consider now a special case: $\alpha(s)=\alpha(t)=n$. In this case we obtain

$$A(s=t,u)=\frac{1}{n^2m}\frac{1}{\left(1-\frac{\alpha(s)}{n}\right)^2}\left[C(l,\alpha(u))+2C(n,\alpha(s))\frac{\left(1-\frac{\alpha(s)}{n}\right)}{\left(1-\frac{\alpha(u)}{l}\right)}\right]$$
$$=\frac{1}{n^2m}\frac{1}{\left(1-\frac{\alpha(s)}{n}\right)^2}[\sin\pi\alpha(u)+2\sin\pi\alpha(s)]\frac{1}{\left(1-\frac{\alpha(u)}{l}\right)}=0 \tag{1.18b}$$

This result is in accordance with that of Ref. [6], where it was obtained differently. The tachyonic case is rather easy to consider now. Indeed, using Eqs. (1.6), (1.14) and (1.15) and taking into account the Veneziano condition, let us assume that, say, $\alpha(s)=0$. Then, in view of symmetry of Eqs. (1.18a) and (1.18b), we need to let $\alpha(t)=0$ as well to check if Eq. (1.18b) holds. This leaves us with the option: $\alpha(u)=-1$. With such a constraint we obtain (since $\Gamma(1)=1$),

$$A(s,t,u)=\frac{\pi}{\alpha(t)}+\frac{\pi}{\alpha(s)}-\frac{\pi}{\alpha(t)}-\frac{\pi}{\alpha(s)}=0,$$

as required. Hence, indeed, even in the tachyonic case, Eq. (1.18b) holds in accordance with earlier results [6]. This means that one cannot observe tachyons in both channels simultaneously. But even to observe them in one channel is unphysical. Moreover, Eq. (1.18b) implies that only situations for which $\alpha(s)\neq\alpha(t)\neq\alpha(u)$ can be in principle physically observable.

By combining the Veneziano condition with such a constraint leaves us with the following options:

(a) $\alpha(s),\alpha(t)>0,\alpha(u)<0$;
(b) $\alpha(s)>0,\alpha(t),\alpha(u)<0$ plus the rest of cyclically permuted inequalities.

This means that not only tachyons of the type $\alpha(s)=0$ (or $\alpha(t)=0$, or $\alpha(u)=0$) could be present but also those for which, for example, $\alpha(s)<0$. This is so because, according to known results for standard open string theory [2] in 26 space–time dimensions, $\alpha(s)=1+\frac{1}{2}s$. When $\alpha(s)=0$, such convention produces the only one tachyon: $s=-2=M^2$, and the whole spectrum (open string) is given by $M^2=-2,0,2,\dots,2n$, where n is a non-negative integer. Incidentally, for the closed bosonic string under the same conditions the spectrum is known to be $M^2=-8,0,8,\dots,8n$. No other masses are permitted. The requirements like those in (a) and (b) produce additional complications however. For instance, let $\alpha(s)=1$

and consider the following option: $\alpha(t) = 3$, so that we should have $\alpha(u) = -5$. This leads us to the tachyon mass: $M^2 = -12$. It is absent in the spectrum of both types of bosonic strings. The emerging apparent difficulty can actually be bypassed somehow due to the following chain of arguments.

Remark 1.2. Consider, for example, the amplitude $V(s, t)$ and let both s and t be non-tachyonic and, of course, $\alpha(s) \neq \alpha(t)$. Then, naturally, $\alpha(u) < 0$ *is* tachyonic. But, when we use Eq. (1.18a), we notice at once that this creates no difficulty since the $\alpha < 0$ condition simply will eliminate the resonance in the respective channels. E.g. if $V(s, t)$ will have poles for both s and t *then the same particle resonances will occur* in $V(s, u)$ and $V(t, u)$ channels so that, *except the case* $\alpha(s) = 0$ leading to the pole with mass $M^2 = -2$, no other tachyonic states will show up as resonances and, hence, they cannot be observed. This argument is important for designing of the bosonic string model but is in apparent violation of the Veneziano condition, Eq. (1.5). It is violated if the tachyons of larger negative mass are *not* present in the spectrum. Since the Veneziano condition is caused by the energy–momentum conservation, it cannot be readily replaced by something else. The arguments just presented explain in part the inadequacy of the existing formulation of the model reproducing the Veneziano amplitudes.

At the same time, irrespective to the hypothetical model one uses for reproduction of these amplitudes, based on the arguments just presented it should be clear that, *effectively*, we have only two possibilities for resonances to be observed. That is experimentally (in view of the Veneziano condition) we can either observe the resonances for combinations $\mathcal{V}_u(s)$ or $\mathcal{V}_u(t)$. Clearly, $\mathcal{V}_u(s) = V(s, t) + V(s, u)$ and $\mathcal{V}_u(t) = V(t, s) + V(t, u)$. Such a conclusion is valid only if we require $V(s, t) = V(t, s)$, etc. It is surely the case for the Veneziano amplitude, Eq. (1.18a). Accordingly, should the Veneziano amplitude be free of tachyons (e.g. $\alpha(s) = 0$), it would be perfectly acceptable. In the light of the results just obtained it can be effectively written as

$$A(s, t, u) = \mathcal{V}_u(s) + \mathcal{V}_u(t). \tag{1.19}$$

The result, Eq. (1.19), survives when, instead of the Veneziano, the tachyon-free Veneziano-like amplitudes are used. These are discussed in Section 3. Mathematical arguments leading to the construction of models associated with such amplitudes are discussed in detail in Parts 2–4 of this work.

To complete this section, we would like to provide a brief preview of arguments leading to reconstruction of these models. Clearly, such a preview is only a small part of other arguments to be discussed. We believe, that the arguments presented below should be especially appealing to readers familiar with string theory.

Following Hirzebruch and Zagier [7] let us consider an identity

$$\begin{aligned}\frac{1}{(1 - tz_0)\cdots(1 - tz_k)} &= (1 + tz_0 + (tz_0)^2 + \cdots)\cdots(1 + tz_n + (tz_n)^2 + \cdots)\\ &= \sum_{n=0}^{\infty}\left(\sum_{k_0+\cdots+k_k=n} z_0^{k_0}\cdots z_k^{k_k}\right)t^n.\end{aligned} \tag{1.20}$$

When $z_0 = \cdots = z_n = 1$, the inner sum in the last expression provides the total number of monomials of the type $z_0^{k_0} \cdots z_n^{k_n}$ with $k_0 + \cdots + k_k = n$. The total number of such monomials is given by the binomial coefficient [8]

$$p(k, n) \equiv \binom{n+k}{k} = \frac{(n+1)(n+2)\cdots(n+k)}{k!}. \tag{1.21}$$

For this special case Eq. (1.20) is converted to a useful expansion,

$$P(k, t) \equiv \frac{1}{(1-t)^{k+1}} = \sum_{n=0}^{\infty} p(k, n) t^n. \tag{1.22}$$

In view of the integral representation of the beta function given by Eq. (1.2), we replace $k + 1$ by $\alpha(s) + 1$ in Eq. (1.22) and use it in the beta function representation of $V(s, t)$ given by Eq. (1.2). Straightforward calculation produces the following result known in string theory [2]:

$$V(s, t) = -\sum_{n=0}^{\infty} p(\alpha(s), n) \frac{1}{\alpha(t) - n}. \tag{1.23}$$

The r.h.s. of Eq. (1.23) can be interpreted as the Laplace transform of the partition function, Eq. (1.22). Such an interpretation is not merely formal. To see this, following Vergne [9], let us consider a region Δ_k of $\mathbf{R}^k$ consisting of all points $v = (t_1, t_2, \ldots, t_k)$, such that coordinates t_i of v are non-negative and satisfy the constraint: $t_1 + t_2 + \cdots + t_k \leq 1$. Clearly, such a restriction is characteristic for the simplex in $\mathbf{R}^k$. Consider now a *dilated* simplex: $n\Delta_k$ for some non-negative integer *n*. The volume of $n\Delta_k$ is easily calculated and is known to be

$$\mathrm{vol}(n\Delta_k) = \frac{n^k}{k!}. \tag{1.24}$$

Next, let us consider points $v = (u_1, u_2, \ldots, u_k)$ with *integral* coordinates *inside* the dilated simplex $n\Delta_k$. The total number of points with integral coordinates inside $n\Delta_k$ is given by $p(k, n)$, Eq. (1.21), i.e.

$$p(k, n) = |n\Delta_k \cap \mathbf{Z}^k| = \frac{(n+1)(n+2)\cdots(n+k)}{k!}. \tag{1.25}$$

The function $p(k, n)$ happens to be the non-negative integer. It arises naturally as the dimension of the quantum Hilbert space associated (through the coadjoint orbit method) with the symplectic manifold of dimension $2k$ constructed by “inflating” Δ_k. Although the details related to such symplectic manifolds and the associated with them dynamical systems will be provided in Parts 2 and 3, the next section supplies additional relevant information.

2. Relationship between the hypergeometric functions and the Veneziano amplitudes

The fact that the hypergeometric functions are the simplest solutions of the Knizhnik–Zamolodchikov equations of CFT is well documented [10]. The connection between these functions and the toric varieties (to be discussed in Part 2) had been also developed in papers by Gelfand, Kapranov and Zelevinsky (GKZ) [11]. Hence, we see no need in duplication of their results in this work. Instead, we would like to discuss other aspects of hypergeometric functions and their connections with the Veneziano amplitudes emphasizing similarities and differences between strings and CFT.

For reader's convenience, we begin by introducing some standard notations. In particular, let

$$(a, n) = a(a+1)(a+2)\cdots(a+n-1)$$

and, more generally, $(a) = (a_1, \dots, a_p)$ and $(c) = (c_1, \dots, c_q)$. With help of these notations the (p, q)-type hypergeometric function can be written as

$$ {}_pF_q[(a);(c);x] = \sum_{n=0}^{\infty} \frac{(a_1, n)\cdots(a_p, n)}{(c_1, n)\cdots(c_q, n)}\frac{x^n}{n!}. \tag{2.1}$$

In particular, the hypergeometric function in the form known to Gauss is just ${}_2F_1 = F[a, b; c; x]$. Practically all elementary functions and almost all special functions can be obtained as special cases of the hypergeometric function just defined [12].

We are interested in connections between the hypergeometric functions and the Schwarz–Christoffel (S–C) mapping problem. The essence of this problem lies in finding a function $\varphi(\zeta) = z$ which maps the upper half plane $\mathrm{Im}\,\zeta > 0$ (or, equivalently, the unit circle) into the exterior of the n-sided polygon located on the Riemann sphere considered as one-dimensional complex projective space $\mathbf{CP}^1$ (i.e. $z \in \mathbf{CP}^1$). Traditionally, the pre-images $a_1, \dots, a_n$ of the polygon vertices located at points $b_1, \dots, b_n$ in $\mathbf{CP}^1$ are placed onto x-axis of ζ-plane so that $\varphi(a_i) = b_i$, $i = 1 - n$. Let the interior angles of the polygon be $\pi\alpha_1, \dots, \pi\alpha_n$, respectively. Then the exterior angles μ_i are defined through relations $\pi\alpha_i + \pi\mu_i = \pi$, $i = 1 - n$. The exterior angles are subject to the constraint: $\sum_{i=1}^{n} \mu_i = 2$. The above data allow us to write for the S–C mapping function the following known expression:

$$\varphi(\zeta) = C\int_0^{\zeta} (t-a_1)^{-\mu_1}\cdots(t-a_n)^{-\mu_n} + C'. \tag{2.2}$$

If one of the points, say a_n, is located at infinity, it can be shown that in the resulting formula for mapping the last term under the integral can be deleted.

Consider now the simplest but relevant example of mapping of the upper half plane into a triangle with angles α, β and γ subject to Euclidean constraint: $\alpha + \beta + \gamma = 1$. Let, furthermore, $a_1 = 0$, $a_2 = 1$ and $a_3 = \infty$. Using Eq. (2.2) (with $C = 1$) we obtain for the length c of the side of the triangle:

$$c=\int_0^1\left|\frac{\mathrm{d}\varphi(\zeta)}{\mathrm{d}\zeta}\,\mathrm{d}\zeta\right|=\int_0^1 z^{\alpha-1}(1-z)^{\beta-1}=B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(1-\gamma)}. \tag{2.3}$$

Naturally, two other sides can be determined in the same way. Much more efficient is to use the familiar elementary trigonometry relation

$$\frac{c}{\sin\pi\gamma}=\frac{b}{\sin\pi\beta}=\frac{a}{\sin\pi\alpha}.$$

Then, using Eq. (1.9a), we obtain for the sides the following results: $c=\frac{1}{\pi}[\sin\pi\gamma]\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)$; $b=\frac{1}{\pi}[\sin\pi\beta]\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)$; $a=\frac{1}{\pi}[\sin\pi\alpha]\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)$. The perimeter length $\mathcal{L}=a+b+c$ of the triangle is just the full Veneziano amplitude, Eq. (1.1). As is well known [13], the conformal mapping with Euclidean constraint $\alpha+\beta+\gamma=1$ can be performed only for three sets of *fixed* angles. Another four sets of angles belong to the spherical case: $\alpha+\beta+\gamma>1$, while countable infinity of angle sets exist for the hyperbolic case: $\alpha+\beta+\gamma<1$. Hence, the associated with such mappings Fuchsian-type equations used in some formulations of string theory will not be helpful in deriving the Veneziano amplitudes. These equations are useful however in the CFT as is well known [10].

It is well documented that, to some extent, development of the bosonic string theory is inseparable from attempts at multidimensional generalization of Euler's beta function [6]. Analogous developments also took place in the theory of hypergeometric functions where they proceeded along two related lines. To illustrate the key ideas, following Deligne and Mostow [14], let us consider the standard hypergeometric function which, up to a constant,[2] is given by

$$F[a,b;c;x]\doteq\int_1^\infty u^{a-c}(u-1)^{c-b-1}(u-x)^{-a}\,\mathrm{d}u. \tag{2.4}$$

The multidimensional (multivariable) analogue of the above function, according to Picard (in notations of Deligne and Mostow), is given by

$$F[x_2,\ldots,x_{n+1}]=\int_1^\infty u^{-\mu_0}(u-1)^{-\mu_1}\prod_{i=2}^{n+1}(u-x_i)^{-\mu_i}\,\mathrm{d}u, \tag{2.5}$$

provided that $x_0=0, x_1=1$, and as before, $\sum_{i=0}^n\mu_i=2$. At the same time, using the alternative representation of $F[a,b;c;x]$ given by

$$F[a,b;c;x]\doteq\int_0^1 z^{b-1}(1-z)^{c-b-1}(1-zx)^{-a}\,\mathrm{d}z \tag{2.6}$$

[2] To indicate this we use symbol $\doteq$.

one obtains as well the following multidimensional generalization:

$$F[\alpha, \beta, \beta', \gamma; x, y] \doteq \iint_{\substack{u\geq 0, v\geq 0\\ u+v\leq 1}} u^{\beta-1} v^{\beta'-1} (1-u-v)^{\gamma-\beta-\beta'-1} (1-ux)^{-\alpha} \times (1-vy)^{-\alpha'}\, \mathrm{d}u\, \mathrm{d}v. \tag{2.7}$$

This result was obtained by Horn at the end of 19th century and was subsequently reanalyzed and extended by GKZ. Looking at the last expression one can design by analogy the multidimensional extension of the Euler's beta function. In view of Eq. (1.2), it is given by the following integral attributed to Dirichlet:

$$\mathcal{D}(x_1, \ldots, x_k) = \iint_{\substack{u_1\geq 0,\ldots,u_k\geq 0\\ u_1+\cdots+u_k\leq 1}} u_1^{x_1-1} u_2^{x_2-1} \cdots u_k^{x_k-1} (1-u_1-\cdots-u_k)^{x_{k+1}-1} \times \mathrm{d}u_1 \cdots \mathrm{d}u_k. \tag{2.8}$$

In this integral let $t = u_1 + \cdots + u_k$. This allows us to use already familiar expansion, Eq. (1.22). In addition, however, we would like to use the following identity:

$$t^n = (u_1 + \cdots + u_k)^n = \sum_{n=(n_1,\ldots,n_k)} \frac{n!}{n_1! n_2! \cdots n_k!} u_1^{n_1} \cdots u_k^{n_k} \tag{2.9}$$

with restriction $n = n_1 + \cdots + n_k$. This type of identity was used earlier in our work on Kontsevich–Witten model [15]. Moreover, from the same paper we obtain the alternative and very useful form of the above expansion

$$(u_1 + \cdots + u_k)^n = \sum_{\lambda \vdash k} f^{\lambda} S_{\lambda}(u_1, \ldots, u_k), \tag{2.10}$$

where the Schur polynomial S_λ is defined by

$$S_\lambda(u_1, \ldots, u_k) = \sum_{n=(n_1,\ldots,n_k)} K_{\lambda,n} u_1^{n_1} \cdots u_k^{n_k} \tag{2.11}$$

with coefficients $K_{\lambda,n}$ known as Kostka numbers [16], f^λ being the number of standard Young tableaux of shape λ and the notation $\lambda \vdash k$ meaning that λ is a partition of k. Through such connection with Schur polynomials one can develop connections with Kadomtsev–Petviashvili (KP) hierarchy of non-linear exactly integrable systems on one hand, and with the theory of Schubert varieties on another [15]. We shall provide more details on such a connection in Part 4. Additional striking similarities between the results of this paper and those of the Kontsevich–Witten model will be discussed in Parts 2–4.

$$A(1, \ldots k) = \frac{\Gamma_{n_1 \ldots n_k}(\alpha(s_{k+1}))}{(\alpha(s_1) - n_1) \cdots (\alpha(s_k) - n_k)}. \tag{2.12}$$

Even though the residue $\Gamma_{n_1 \ldots n_k}(\alpha(s_{k+1}))$ contains all the combinatorial factors, the obtained result should still be symmetrized (in accord with the 4-particle case considered by Veneziano) in order to obtain the full multiparticle Veneziano amplitude. Since in such general multiparticle case the same expansion, Eq. (1.22), was used, arguments of the previous section can be applied to the present case as well, thus leading to the same model considered by Vergne [9]. Details will be discussed in Parts 2 and 3.

3. Veneziano amplitudes from Fermat hypersurfaces

3.1. General considerations

In 1967, a year earlier than Veneziano's paper was published, the paper [17] by Chowla and Selberg had appeared relating Euler's beta function to the periods of elliptic integrals. The result by Chowla and Selberg was generalized by Weil whose two influential papers [18,19] have brought into picture the periods of Jacobians of the Abelian varieties, Hodge rings, etc. Being motivated by these papers, Gross had written a paper [20] in which the beta function appears as a period associated with the differential form "living" on the Jacobian of the Fermat curve. His results as well as those by Rohrlich (placed in the appendix to Gross paper) have been subsequently documented in the book by Lang [21]. Although in the paper by Gross [20] the multidimensional extension of the beta function is briefly considered, e.g. read p. 207 of Ref. [20], the computational details were not provided, however. We provide these details below following some ideas developed in lecture notes by Deligne [22]. To obtain the multidimensional extension of the beta function following logic of the paper by Gross, one needs to replace the Fermat curve by the Fermat hypersurface, to embed it into the projective space and, by complexification, to treat it as the Kähler manifold. The differential forms living on such a manifold are associated with periods of Fermat hypersurface. In Parts 2 and 3 of this work we shall argue that thus obtained Kähler manifold is of Hodge-type. We will also provide arguments independent from those by Weil [18,19] needed to arrive at the same conclusions. In his lecture notes Deligne noticed that the Hodge theory requires some essential changes (e.g. mixed Hodge structures, etc.) if the Hodge–Kähler manifold possess singularities. Such modifications may be needed upon development of the formalism we are about to discuss. A monograph by Carson et al. [23] contains an up to date exhaustive information regarding such modifications, etc. Fortunately, to obtain the multiparticle Veneziano amplitudes, such complications are not necessary. Hence we proceed directly with description of main ideas.

To illustrate these ideas, in accordance with Ref. [23], we are following the arguments by Griffiths [24]. To this purpose, let us begin with the simplest example of calculation of the following period integral:

$$\pi(\lambda) = \oint_{\Gamma} \frac{\mathrm{d}z}{z(z-\lambda)} \tag{3.1}$$

taken along the closed contour Γ in the complex z-plane. Since this integral depends upon parameter λ the period $\pi(\lambda)$ is some function of λ. It can be determined by straightforward differentiation of $\pi(\lambda)$ with respect to λ thus leading to the desired differential equation

$$\lambda\pi'(\lambda) + \pi(\lambda) = 0 \tag{3.2}$$

enabling us to calculate $\pi(\lambda)$. This simple result can be vastly generalized to cover the case of period integrals of the type

$$\Pi(\lambda) = \oint_{\Gamma} \frac{P(z_1, \dots, z_n)}{Q(z_1, \dots, z_n)} \, \mathrm{d}z_1 \wedge \mathrm{d}z_2 \cdots \wedge \mathrm{d}z_n. \tag{3.3}$$

The equation $Q(z_1, \dots, z_n) = 0$ determines algebraic variety. It may contain a parameter (or parameters) λ so that the polar locus of values of z's satisfying equation $Q = 0$ depends upon this parameter(s). By analogy with Eq. (3.2), it is possible to obtain a set of differential equations of P–F type. This was demonstrated originally by Manin [25]. In this work we are not going to develop this line of research, however. Instead, following Griffiths [24], we want to analyze in some detail the nature of the expression under the integral sign in Eq. (3.3).

If $\mathbf{x} = (x_0, \dots, x_n)$ are homogenous coordinates of a point in *projective space* and $z = (z_1, \dots, z_n)$ are the associated coordinates of the point in the *affine space*, where $z_i = \frac{x_i}{x_0}$, then the rational n-form ω is given in the *affine* space by

$$\omega = \frac{P(z_1, \dots, z_n)}{Q(z_1, \dots, z_n)} \, \mathrm{d}z_1 \wedge \mathrm{d}z_2 \cdots \wedge \mathrm{d}z_n \tag{3.4}$$

with rational function $\frac{P}{Q}$ being a quotient of two homogenous polynomials of the *same* degree. Upon substitution: $z_i = \frac{x_i}{x_0}$, the form $\mathrm{d}z_1 \wedge \mathrm{d}z_2 \cdots \wedge \mathrm{d}z_n$ changes to

$$\mathrm{d}z_1 \wedge \mathrm{d}z_2 \cdots \wedge \mathrm{d}z_n = (x_0)^{-(n+1)} \sum_{i=0}^{n} (-1)^i x_i \, \mathrm{d}x_0 \wedge \cdots \wedge \mathrm{d}\hat{x}_i \wedge \cdots \wedge \mathrm{d}x_n,$$

where the hat on the top of x_i means that it is excluded from the product. It is convenient now to define the form ω_0 via

$$\omega_0 := \sum_{i=0}^{n} (-1)^i x_i \, \mathrm{d}x_0 \wedge \cdots \wedge \mathrm{d}\hat{x}_i \wedge \cdots \wedge \mathrm{d}x_n$$

so that in terms of *projective space* coordinates the form ω can be rewritten as

$$\omega = \frac{p(\mathbf{x})}{q(\mathbf{x})} \omega_0, \tag{3.5}$$

where $p(\mathbf{x}) = P(\mathbf{x})$ and $q(\mathbf{x}) = Q(\mathbf{x}) x_0^{n+1}$, or in more symmetric form, $q(\mathbf{x}) = Q(\mathbf{x}) x_0 \cdots x_n$. In this case the degree of the denominator of the rational function $\frac{p}{q}$ is that of the numerator $+(n+1)$. This is the result of Corollary 2.11 of Griffiths paper [24]. Conversely, each homogenous differential form ω in projective space can be written in affine space upon substitution: $x_0 = 1$ and $x_i = z_i$, $i \neq 0$.

We would like to take advantage of this fact now. To this purpose, as an example, we would like to study the period integrals associated with equation describing Fermat hypersurface in complex projective space

$$\mathcal{F}(N) : x_0^N + \cdots + x_n^N + x_{n+1}^N = 0. \tag{3.6}$$

We would like to consider the set of independent linear forms $x_i^{\langle c_i \rangle}$, $i = 0 - (n+1)$, where $\langle c_i \rangle$ denotes representative of c_i in $\mathbf{Z}$ such that for now $1 \leq \langle c_i \rangle \leq N - 1$.[3] They can be interpreted as the set of hyperplanes in $\mathbf{C}^{n+1}$ whose complement is the complex algebraic torus T as will be explained in detail in Parts 2 and 3. We want to consider the form ω living at the intersection of T with $\mathcal{F}$. To this purpose it is convenient to introduce the average $\langle c \rangle$ as follows:

$$\langle c \rangle = \frac{1}{N} \sum_i \langle c_i \rangle. \tag{3.7}$$

The numbers c_i belong to the set $X(S^1)$ given by

$$X(S^1) = \left\{ \bar{c} \in \left(\frac{\mathbf{Z}}{N\mathbf{Z}} \right)^{n+2} \equiv \left(\frac{\mathbf{Z}}{N\mathbf{Z}} \right) \times \cdots \times \left(\frac{\mathbf{Z}}{N\mathbf{Z}} \right) \middle| \bar{c} \right.$$
$$\left. = (c_0, \ldots, c_{n+1}), \sum_i c_i = 0 \bmod N \right\}. \tag{3.8}$$

The true meaning of the condition $\sum_i c_i = 0 \bmod N$ is illustrated below by using the Fermat hypersurface $\mathcal{F}(N)$ as an example. In this case the form ω, Eq. (3.5), is given by

$$\omega = \frac{x_0^{\langle c_0 \rangle - 1} \cdots x_{n+1}^{\langle c_{n+1} \rangle - 1}}{(x_0^N + \cdots + x_n^N + x_{n+1}^N)^{\langle c \rangle}} \omega_0. \tag{3.9}$$

By design, it satisfies all of the requirements of Corollary 2.11 of Griffiths paper.

3.2. The 4-particle Veneziano-like amplitude

Using Eq. (3.9), let us consider the simplest but important case: $n = 1$. It is relevant for calculation of 4-particle Veneziano-like amplitude. Converting ω into affine form according to Griffiths prescription we obtain the following result for the period integral:

$$I_{\text{aff}} = \oint_\Gamma \frac{1}{x_1^N + x_2^N \mp 1} \, \mathrm{d}x_1^{\langle c_1 \rangle} \wedge \mathrm{d}x_2^{\langle c_2 \rangle}. \tag{3.10a}$$

The $\pm$ sign in the denominator requires some explanation. Indeed, let us for a moment restore the projective form of this integral. By doing so, we obtain the following integral:

$$I_{\text{proj}} = \oint_\Gamma \frac{z_1^{\langle c_1 \rangle} z_2^{\langle c_2 \rangle} z_0^{\langle c_0 \rangle}}{z_1^N + z_2^N \pm z_0^N} \left(\frac{\mathrm{d}z_1}{z_1} \wedge \frac{\mathrm{d}z_2}{z_2} - \frac{\mathrm{d}z_0}{z_0} \wedge \frac{\mathrm{d}z_2}{z_2} + \frac{\mathrm{d}z_0}{z_0} \wedge \frac{\mathrm{d}z_1}{z_1} \right). \tag{3.10b}$$

[3] These limits for $\langle c_i \rangle$ are in accordance with Gross [20, p. 198]. Subsequently, they will be changed below to $1 \leq \langle c_i \rangle \leq N$.

It is manifestly symmetric with respect to permutation of its arguments by construction. In addition, we would like to be invariant with respect to scale transformations of the type: $z_j \to z_j\xi^j$, where $\xi^j = \exp\left(\pm\mathrm{i}\frac{2\pi j}{N}\right)$ with $1 \leq j \leq N-1$. Such scaling is used extensively in the theory of invariants of the pseudo-reflection groups. Its meaning will be discussed in Part 2 in connection with invariance properties of the Veneziano and Veneziano-like amplitudes. For now, it is sufficient to realize only that the numerator of the integrand in Eq. (3.10b) as a whole acquires the following phase factor: $\exp\left\{\mathrm{i}\frac{2\pi}{N}(\langle c_1\rangle j + \langle c_2\rangle k + \langle c_0\rangle l)\right\}$. Since by design the integral I_{proj} is made to satisfy Corollary 2.11 discussed in the previous subsection, it is sufficient to require

$$\langle c_1\rangle j + \langle c_2\rangle k + \langle c_0\rangle l = N \tag{3.11a}$$

in order to make it manifestly scale invariant (torus action invariant in the terminology of Part 2). We shall call Eq. (3.11a) the "Veneziano condition" while Eq. (3.11b) we shall call the "Shapiro–Virasoro" condition.[4] Transition from the projective to affine space breaks the permutational symmetry firstly because of selecting, say, z_0 (and requiring it to be 1) and, secondly, by possibly switching the sign in front of z_0. The permutational symmetry can be restored in the style of Veneziano, e.g. see Eq. (1.1). The problem of switching the sign in front of z_0 can be treated similarly but requires extra care. This is so because instead of the factor ξ^j used above we could have used ε^j, where $\varepsilon = \exp\left(\pm\mathrm{i}\frac{\pi}{N}\right)$[5] Use of such a factor makes the integral I_{proj} also torus action invariant. But for this case the condition, Eq. (3.11a), has to be changed into

$$\langle c_1\rangle j + \langle c_2\rangle k + \langle c_0\rangle l = 2N \tag{3.11b}$$

in accordance with Lemma 1 of Gross [20]. By such a change we are in apparent disagreement with Corollary 2.11 by Griffiths. We write "apparent" because, fortunately, there is a way to reconcile Corollary 2.11 by Griffiths with Lemma 1 by Gross. It will be discussed below. Already assuming that this is the case, we notice that there are at least two different classes of transformations leaving I_{proj} unchanged. When switching to the affine form these two classes are not equivalent: the first leads to differential forms of the first kind while the second to that of the second kind [20,21]. Both are living on the Jacobian variety $J(N)$ associated with the Fermat surface $\mathcal{F}(N) : z_1^N + z_2^N \pm 1 = 0$. It happens that physically more relevant are the forms of the second kind. We would like to describe them now.

We begin by noticing that in switching from the projective to affine space the following set of $3N$ points (at infinity) should be deleted from the Fermat curve $z_1^N + z_2^N + z_3^N = 0$. These are: $(\varepsilon\xi^j, 0, 1)$, $(0, \varepsilon\xi^j, 1)$, $(\varepsilon^2\xi^j, \varepsilon\xi^j, 0)$, respectively [26]. By assuming that this is the case and parameterizing z_1 and z_2 as $z_1 = \varepsilon t_1^{1/N}$ and $z_2 = \varepsilon t_2^{1/N}$, we obtain the simplex equation $t_1 + t_2 = 1$ as deformation retract for $\mathcal{F}(N)$.[6] After this, Eq. (3.10a) acquires the

[4] These names are given by analogy with the existing terminology for the open (Veneziano) and closed (Shapiro–Virasoro) bosonic strings. Clearly, in the present context they emerge for reasons different from those used in conventional formulations.

[5] Both options will be explained in Part 2 from the point of view of the concept of the torus action. For alternative point of view, please read Ref. [26].

[6] The rationale for such substitutions is explained in Part 2.

following form:

$$I_{\text{aff}} = \xi^{j\langle c_1\rangle + k\langle c_2\rangle}\frac{1}{N^2}\oint_{\Gamma}\frac{\varepsilon^{\langle c_1\rangle}t_1^{\langle c_1\rangle/N}\varepsilon^{\langle c_2\rangle}t_2^{\langle c_2\rangle/N}}{t_1+t_2-1}\frac{\mathrm{d}t_1}{t_1}\wedge\frac{\mathrm{d}t_2}{t_2}. \tag{3.12}$$

The overall phase factor guarantees the linear independence of the above period integrals [21] in view of the well-known result: $1+\xi^r+\xi^{2r}+\cdots+\xi^{(N-1)r}=0$. It will be omitted for brevity in the rest of our discussion.

To calculate I_{aff} we need to use generalization of the method of residues for multidimensional complex integrals as developed by Leray [27] and discussed in physical context by Hwa and Teplitz [28] and others. From this reference we find that taking the residue can be achieved either by dividing the differential form in Eq. (3.12) by $\mathrm{d}s = t_1\,\mathrm{d}t_1 + t_2\,\mathrm{d}t_2$ or, equivalently, by writing instead of Eq. (3.12) the following physically suggestive result:

$$I_{\text{aff}} = \frac{1}{N^2}\oint_{\Gamma}\varepsilon^{\langle c_1\rangle}t_1^{\langle c_1\rangle/N}\varepsilon^{\langle c_2\rangle}t_2^{\langle c_2\rangle/N}\frac{\mathrm{d}t_1}{t_1}\wedge\frac{\mathrm{d}t_2}{t_2}\delta(t_1+t_2-1) \tag{3.13}$$

to be discussed further in Parts 2 and 3. For the time being, taking into account that $t_2 = 1-t_1$, after calculating the Leray residue we obtain

$$I_{\text{aff}} = \frac{1}{N^2}\int_0^1 u^{\langle c_1\rangle/N-1}(1-u)^{\langle c_2\rangle/N-1}\,\mathrm{d}u = \frac{1}{N^2}B(a,b), \tag{3.14}$$

where $B(a,b)$ is Euler's beta function (as in Eq. (1.2)) with $a=\frac{\langle c_1\rangle}{N}$ and $b=\frac{\langle c_2\rangle}{N}$. The phase factors had been temporarily suppressed for the sake of comparison with the results of Rohrlich [20] (published as an appendix to the paper by Gross and also discussed in the book by Lang [21]). To make such a comparison, we need to take into account the multivaluedness of the integrand above if it is considered in the standard complex plane. Referring our readers to Chapter 5 of Lang's book [21] allows us to avoid rather long discussion about the available choices of integration contours. Proceeding in complete analogy with the case considered by Lang, we obtain the period $\Omega(a,b)$ of the differential form $\omega_{a,b}$ of the *second* kind living on $J(N)$:

$$\frac{\Omega(a,b)}{N} = \frac{1}{N}\oint_{\Gamma}\omega_{a,b} = \frac{1}{N^2}(1-\varepsilon^{\langle c_1\rangle})(1-\varepsilon^{\langle c_2\rangle})B(a,b). \tag{3.15}$$

The Jacobian $J(N)$ is related to the Fermat curve $\mathcal{F}(N)$ considered as the Riemann surface of genus $g=\frac{1}{2}(N-1)(N-2)$. Obtained result differs from that by Rohrlich only by phase factors: ε's instead of ξ's. The number of such periods is determined by the inequalities of the type $1 \le \langle c_i\rangle \le N-1$. In addition to the differential forms of the second kind, there are also the differential forms of the *third* kind living on $\mathcal{F}(N)$. They can be easily obtained from that of the second kind by relaxing the condition $1\le\langle c_i\rangle\le N-1$ to $1\le\langle c_i\rangle\le N$, Lang [21, p. 39]. The differential forms of the second kind are associated with the de Rham cohomology classes $H^1_{\text{DR}}(\mathcal{F}(N),\mathbf{C})$ [20, Lemma 1]. The differential forms of the first kind, discussed in the book by Lang [21], by design do not have any poles while the differentials of the second kind by design do not have residues. Only differentials of the third kind have poles of order ≤ 1 with non-vanishing residues and, hence, are physically interesting. We shall be dealing mostly with differentials of the second kind converting them eventually into that of the third kind. The differentials of the third kind are linearly independent from that of the first kind according to Lang [21].

Symmetrizing our result, Eq. (3.15), in the spirit of Veneziano ideas we obtain the 4-particle Veneziano-like amplitude

$$A(s,t,u)=\tilde{V}(s,t)+\tilde{V}(s,u)+\tilde{V}(t,u), \tag{3.16}$$

where, for example, upon analytical continuation $V(s,t)$ is given by

$$\tilde{V}(s,t)=\left(1-\exp\left(\mathrm{i}\frac{\pi}{N}(-\alpha(s))\right)\right)\left(1-\exp\left(\mathrm{i}\frac{\pi}{N}(-\alpha(t))\right)\right)B\left(\frac{-\alpha(s)}{N},\frac{-\alpha(t)}{N}\right), \tag{3.17}$$

provided that we have identified $\langle c_i\rangle$ with $\alpha(i)$, etc. Naturally, in arriving at Eq. (3.17) we have extended the differential forms from those of the second kind to those of the third. The analytical properties of such designed Veneziano-like amplitudes are discussed in detail below in subsections on multiparticle amplitudes

3.3. Connection with CFT through hypergeometric functions and the Kac–Moody–Bloch–Bragg condition

In the light of just obtained results, we would like now to compare the hypergeometric function, Eq. (2.6), with the beta function. Taking into account that [12]

$$(1-zx)^{-a}=\sum_{n=0}^{\infty}\frac{(a,n)}{n!}(zx)^n,$$

Eq. (2.6) can be rewritten as follows:

$$F[a,b;c;x]\doteq\sum_{n=0}^{\infty}\frac{(a,n)}{n!}x^n\int_0^1 z^{b+n-1}(1-z)^{c-b-1}\,\mathrm{d}z=\sum_{n=0}^{\infty}\frac{(a,n)}{n!}x^n B(b+n,c-b). \tag{3.18}$$

This result is to be compared with Eq. (3.14). To this purpose it is convenient to rewrite Eq. (3.14) in the following more general form (up to a constant factor):

$$I(m,l)\doteq\int_0^1 u^{\langle c_1\rangle-N+mN/N}(1-u)^{\langle c_2\rangle-N+lN/N}\,\mathrm{d}u=B(a+m,b+l),$$

where $m,l=0,\pm1,\pm2,\ldots$. It is clear that the phase factors entering into Eq. (3.15) will either remain unchanged or will change sign upon such replacements. At the same time the Veneziano condition, Eq. (3.11a), will change into

$$\langle c_0\rangle+\langle c_1\rangle+\langle c_2\rangle=N+mN+lN+kN. \tag{3.19}$$

This result can be explained physically with the help of some known facts from solid state physics, e.g. read Ref. [29]. To this purpose let us consider the result of torus action on the form ω, Eq. (3.9). If we demand this action to be torus action invariant (as it is explained in Part 2), then we obtain

$$\sum_i \langle c_i\rangle m_i = 0 \bmod N \tag{3.20}$$

with m_i being some integers. In particular, consider Eq. (3.20) for a special case of 4-particle Veneziano amplitude. Then, in accordance with the discussion following Eq. (1.5), the Veneziano condition can be rewritten as

$$\langle c_0\rangle m_0 + \langle c_1\rangle m_1 + \langle c_2\rangle m_2 = 0 \bmod N. \tag{3.21}$$

But, in view of the Griffiths Corollary 2.11, the condition mod N (or mod $2N$) for the Veneziano amplitudes should actually be replaced by N (or $2N$). At the same time for the hypergeometric functions in view of Eqs. (3.19) and (3.21), we should write instead

$$\langle c_0\rangle m_0 + \langle c_1\rangle m_1 + \langle c_2\rangle m_2 = mN + lN + kN. \tag{3.22}$$

Such a condition is known in solid state physics as the Bragg equation [29]. This equation plays the central role in determining crystal structure by X-ray diffraction. Lattice periodicity reflected in this equation affects kinematics of scattering processes for phonons and electrons in crystals. Under these circumstances the concepts of particle energy and momentum lose their usual meaning and should be amended to account for the lattice periodicity. The same type of amendments should be made when comparing elementary scattering processes in CFT against those in high energy physics. We shall call Eq. (3.22) the *Kac–Moody–Bloch–Bragg* (*K–M–B–B*) *equation*. In the group-theoretic language of Parts 2 and 3 the difference between the high energy scattering processes and those in CFT is of the same nature as the difference between the Coxeter–Weyl (pseudo)reflection groups and their affine generalizations [30]. The same difference will be explained from the point of view of symplectic and complex manifolds in Part 3.

3.4. Analytical properties of the multiparticle Veneziano and Veneziano-like amplitudes (general considerations)

By analogy with the 4-particle case, the Fermat variety $\mathcal{F}_{\mathrm{aff}}(N)$ in the affine form in the multiparticle case is given by the following equation:

$$\mathcal{F}_{\mathrm{aff}}(N) : Y_1^N + \cdots + Y_{n+1}^N = 1, \quad Y_i = \frac{x_i}{x_0} \equiv z_i. \tag{3.23}$$

As before, use of parameterization $f : z_i = t_i^{1/N} \exp(\pm \frac{\pi \mathrm{i}}{N})$ such that $\sum_i t_i = 1$ allows us to reduce the Fermat variety $\mathcal{F}_{\mathrm{aff}}(N)$ to its deformation retract which is $n+1$ simplex Δ. That is, $f(\mathcal{F}_{\mathrm{aff}}(N)) = \Delta$, where $\Delta : \sum_i t_i = 1$. The period integrals of the type given by Eq. (3.3) with ω form defined by Eq. (3.9) after taking the Leray-type residue are reduced to the following standard form (up to a constant):

$$I \doteq \int_\Delta t_1^{\langle c_1\rangle/N-1} \cdots t_{n+1}^{\langle c_{n+1}\rangle/N-1} \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n, \tag{3.24}$$

where, again, all phase factors have been suppressed temporarily. The important group-theoretic meaning of the integrand in the above integral leading to recovery of the model associated with such integral will be discussed at length in both Parts 2 and 3 of this work.

For $n = 1$ this integral coincides with that given by Eq. (3.14) (up to a constant) as required. As part of preparations for calculation of this integral for $n > 1$ let us first have another look at the case $n = 1$, where we have integrals of the type

$$I = \int_0^1 \mathrm{d}x x^{a-1}(1-x)^{b-1} = B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Alternatively, we can look at

$$\Gamma(a+b)I = \int_0^\infty \int_0^\infty \mathrm{d}x_1\,\mathrm{d}x_2\, x_1^{a-1}x_2^{b-1}\exp(-x_1-x_2). \tag{3.25}$$

In the double integral on the r.h.s. let us consider change of variables: $x_1 = \hat{x}_1 t$, $x_2 = \hat{x}_2 t$ so that $x_1 + x_2 = t$, provided that $\hat{x}_1 + \hat{x}_2 = 1$. Taking t and $\hat{x}_1$ as new variables and taking into account that the Jacobian of such transformation is one, the following result is obtained:

$$\Gamma(a+b)I = \int_0^\infty \mathrm{d}t\, t^{a+b-1}\exp(-t)\int_0^1 \mathrm{d}\hat{x}_1\, \hat{x}_1^{a-1}(1-\hat{x}_1)^{b-1},$$

as expected. Going back to the original integral, Eq. (3.24), and introducing notations $a_i = \frac{\langle c_i\rangle}{N}$ we obtain,

$$\Gamma\left(\sum_{n=1}^{n+1} a_i\right) I \doteq \int_0^\infty \frac{\mathrm{d}t}{t} t^{\sum_{i=1}^{n+1} a_i}\exp(-t)\int_\Delta t_1^{a_1-1}\cdots t_{n+1}^{a_{n+1}-1}\,\mathrm{d}t_1\wedge\cdots\wedge \mathrm{d}t_n. \tag{3.26}$$

By analogy with the case $n = 1$ we introduce new variables: $s_i = tt_i$. Naturally, we expect $\sum_{n=1}^{n+1} s_i = t$ since t_i variables are subject to the simplex constraint $\sum_{i=1}^{n+1} t_i = 1$. With such replacements we obtain

$$\Gamma\left(\sum_{n=1}^{n+1} a_i\right) I = \int_0^\infty \cdots \int_0^\infty \exp\left(-\sum_{n=1}^{n+1} s_i\right) s_1^{a_1}\cdots s_{n+1}^{a_{n+1}}\frac{\mathrm{d}s_1}{s_1}\wedge\cdots\wedge\frac{\mathrm{d}s_{n+1}}{s_{n+1}}$$
$$= \Gamma(a_1)\cdots\Gamma(a_{n+1}).$$

Using this result, the n-particle contribution to the Veneziano amplitude is given finally by the following expression:

$$I \doteq \frac{\prod_{i=1}^{n+1}\Gamma(a_i)}{\Gamma\left(\sum_{n=1}^{n+1} a_i\right)}. \tag{3.27}$$

Remark 3.1. Eq. (3.27) can be found in the paper by Gross [20, p. 206], where it is suggested (postulated) without derivation. Eq. (3.27) provides a complete explicit calculation of the Dirichlet integral, Eq. (2.8), and as such, can be found, for example, in the book by Edwards

[31] published in 1922. Calculations similar to ours also can be found in lecture notes by Deligne [22]. We shall use some additional results from his notes below.

Our calculations are far from being complete however. To complete our calculations we need to introduce the appropriate phase factors. In addition, we need to discuss carefully the analytic continuation of just obtained expression for amplitude to negative values of parameters a_i. Fortunately, the phase factors can be reinstalled in complete analogy with the 4-particle case in view of the following straightforwardly verifiable identity:

$$B(x,y)B(x+y,z)B(x+y+z,u)\cdots B(x+y+\cdots+t,l)=\frac{\Gamma(x)\Gamma(y)\cdots\Gamma(l)}{\Gamma(x+y+\cdots+l)}. \tag{3.28}$$

Because of this identity, the multiphase problem is reduced to that we have considered already while looking at the 4-particle case and, hence, can be considered as solved. The analytic continuation problem connected with the multiphase problem is much more delicate and requires longer explanations.

The first difficulty we encounter is related to the constraints imposed on $\langle c_i\rangle$ factors discussed in connection with the 4-particle case, e.g. restriction: $1\le\langle c_i\rangle\le N-1$ (or $1\le\langle c_i\rangle\le N$). To resolve this difficulty, we shall follow Deligne's lecture notes [22]. We begin with Eq. (3.9). The Veneziano condition, Eq. (3.11a), extended to the multivariable case is written as

$$1=\langle c\rangle=\frac{1}{N}\sum_i\langle c_i\rangle, \tag{3.29}$$

whereas Corollary 2.11 by Griffiths *does not* require this constraint to be imposed. To satisfy this corollary, it is sufficient for us to require only $m=\langle c\rangle$ for some integer m to be specified below. Clearly, such a requirement will change the total sum of exponents accordingly in the numerator of Eq. (3.9). In particular, for $m=2$ we would reobtain Eq. (3.11b). It should be noted at this point that Lemma 1 by Gross [20] although imposes such a constraint but was actually proven *not* in connection with the period differential form, Eq. (3.9). This lemma implicitly assumes that the Leray residue *was taken already* and deals with the differential forms occurring as a result of such operation. To avoid guessing in the present case, we need to initiate our analysis again starting from Eq. (3.9) and taking into account Corollary 2.11.

Following Deligne [22], let us discuss what happens if we replace $\langle c\rangle$, Eq. (3.7), by $\langle -c\rangle$. In view of definition of the bracket sign $\langle\cdot\rangle$ we obtain

$$\langle -c\rangle=\frac{1}{N}\sum_i\langle -c_i\rangle=\frac{1}{N}\sum_i\langle -c_i+N\rangle=n+2-\langle c\rangle, \tag{3.30}$$

where the factor $n+2$ comes from the sum $\sum_i 1$ and $\langle c\rangle$ is the same as in Eq. (3.7), provided that $1\le\langle c_i\rangle\le N$. This result implies that the number m defined above can be only in the range

$$\frac{n+2}{N}\le m\le n+2. \tag{3.31}$$

Remark 3.2. The Fermat variety $\mathcal{F}(N)$, Eq. (3.6), is of the *Calabi–Yau* type if and only if $n + 2 = N$ [32, p. 531]. Clearly, this requirement is equivalent to the Veneziano condition, Eq. (3.29), i.e. $m = 1$.

Remark 3.3. By *not* imposing this condition we can still get many physically relevant and interesting results using Deligne's lecture notes [22]. We have encountered this already while arriving at Eq. (3.11b). Clearly, this equation is anyway reducible to Eq. (3.11a) but earlier we obtained physically important phase factor ε (instead of ξ) by working with Eq. (3.11b). It should be obvious by now that m is responsible for change in phase factors: from ξ (for $m = 1$) to ε (for $m = 2$) to $\hat{\varepsilon}_m = \exp(\mathrm{i}\frac{2\pi}{mN})$ (for $m > 2$). Physical significance of these phase factors is discussed in the next subsection.

To extend these results we need to introduce several new notations now. Let $V_{\mathbf{C}}$ be a finite dimensional vector space over $\mathbf{C}$. A $\mathbf{C}$-rational Hodge structure of weight n on V is a decomposition $V_{\mathbf{C}} = \bigoplus_{p+q=n} V^{p,q}$, such that $\overline{V}^{p,q} = V^{q,p}$. We extend the definition of the torus action (to be given rigorously in Part 2) in order to accommodate the complex conjugation: $(t, V^{p,q}) = t^{-p}\bar{t}^{-q} V^{p,q}$. Next, we define the filtration (the analog of the flag decomposition, e.g. see Ref. [33] or our earlier work, Ref. [15], for rigorous definitions and further details) via $F^p V = \bigoplus_{p'>p} V^{p',q'}$ so that $\cdots \supset F^p V \supset F^{p+1} V \supset \cdots$ is a decreasing filtration on V. The differential form, Eq. (3.9), belongs to the space $\Omega_m^{n+1}(\mathcal{F})$ of differential forms such that $\omega = \frac{p(\mathbf{z})}{q(\mathbf{z})^m}\omega_0$, where $p(\mathbf{z})$ is a homogenous polynomial of degree $m\deg(q) - (n + 2)$. Such differential forms have a pole of order $\leq m$. As in the standard complex analysis, one can define the multidimensional analogue of the residue via map $R(\omega) : \Omega_m^{n+1}(\mathcal{F}) \to H^n(\mathcal{F}, \mathbf{C})$ via

$$\langle \sigma, R(\omega)\rangle = \frac{1}{2\pi \mathrm{i}} \int_{\sigma} \omega, \sigma \in H_n(\mathcal{F}, \mathbf{C}). \tag{3.32}$$

Deligne proves that:

(a) $H^n(\mathcal{F}, \mathbf{C}) = \bigoplus_{\bar{c}\neq 0} H^n(\mathcal{F}, \mathbf{C})_{\bar{c}}$, where
(b) $H^n(\mathcal{F}, \mathbf{C})_{\bar{c}} \subset F^{\langle c\rangle -1} H^n(\mathcal{F}, \mathbf{C})$, while the complex conjugate of $H^n(\mathcal{F}, \mathbf{C})_{\bar{c}}$ is given by $H^n(\mathcal{F}, \mathbf{C})_{-\bar{c}} \subset F^{n-\langle c\rangle +1} H^n(\mathcal{F}, \mathbf{C})$.

Thus, by construction, $\mathrm{H}^n(\mathcal{F}, \mathbf{C})_{\bar{c}}$ is of bidegree (p, q) with $p = \langle c\rangle - 1, q = n - p$, while its complex conjugate $H^n(\mathcal{F}, \mathbf{C})_{-\bar{c}}$ is of bidegree (q, p). Obtained cohomologies are non-trivial and of Hodge-type only when $\langle c\rangle \neq 1$. Finally, the procedure of extracting the residue from the integral in Eq. (3.32) with ω containing a pole of order m is described in the book by Hwa and Teplitz [28] and in spirit is essentially the same as in the standard one-variable complex analysis. Therefore, after all, we end up again with the differential form ω, Eq. (3.9), with $\langle c\rangle = 1$. However, this form will be used with the phase factors $\hat{\varepsilon}_m$ instead of ξ. Physical consequences of this replacement are considered in the next subsection.

3.5. Analytical properties of the Veneziano-like amplitudes (ramifications)

Earlier obtained results of this section allow us to write the following 4-particle Veneziano-like amplitude,

$$A(s,t,u)=\tilde{V}(s,t)+\tilde{V}(s,u)+\tilde{V}(t,u),$$

where, for instance,

$$\tilde{V}(s,t)=\left(1-\exp\left(\mathrm{i}\frac{\pi}{N}(-\alpha(s))\right)\left(1-\exp\left(\mathrm{i}\frac{\pi}{N}(-\alpha(t))\right)\right)B\left(-\frac{\alpha(s)}{N},-\frac{\alpha(t)}{N}\right)\right). \tag{3.33}$$

Although this result was obtained by the same analytic continuation as in the case of the Veneziano amplitude, the resulting analytical properties of such Veneziano-like amplitude are markedly different. In this section we would like to discuss these important differences.

We begin by noticing that, in view of Eq. (3.11b), the Veneziano condition in its simplest form: $a+b+c=1$, upon analytic continuation, leads again to the requirement: $\alpha(s)+\alpha(t)+\alpha(u)=-1$, if we identify, for example, $\frac{\langle c_1\rangle}{N}=a_1$ with $-\alpha(s)$, etc. This naive identification leads to some difficulties, however. Indeed, since physically we are interested in the poles and zeros of gamma functions, we expect our parameters a, b and c to be integers. This is possible only if the absolute values of $\langle c_1\rangle$, $\langle c_2\rangle$ and $\langle c_3\rangle$ are greater than or equal to N. By allowing these parameters to become greater than N we would formally violate the requirements of Corollary 2.11 by Griffiths, e.g. see Eq. (3.9) and the discussion around it. Fortunately, the occurring difficulty can by resolved. For instance, one can postulate Eqs. (3.16) and (3.33) as *defining relations* for the Veneziano-like amplitudes as it was done historically by Veneziano for what has become known as the Veneziano amplitudes. In this case one is confronted with the problem of finding some physical model reproducing such amplitudes. To facilitate search for such a model, it is reasonable to impose the same constraints as for the standard Veneziano amplitudes. Clearly, if we want to use earlier obtained results, we must, in addition to these constraints, to impose the constraint coming from Corollary 2.11. This corollary formally forbids us from consideration of ratios $\frac{\langle c_i\rangle}{N}$, whose absolute value is greater than 1 as we have discussed in the previous subsection. After a short while of thinking, this complication creates no additional problems, however. This can be seen already in the example of Eq. (1.14) for $\sin\pi x$. Indeed, consider the function

$$F(x)=\frac{1}{\sin\pi x}.$$

It will have the first order poles for $x=0,\pm1,\pm2,\ldots$. If we formally define the bracket operator $\langle\cdots\rangle$ by analogy with that defined before Eq. (3.7), e.g. $0<\langle x\rangle\leq 1\,\forall x$, then to reproduce the poles of $F(x)$ it is sufficient to write

$$F(x)=\frac{1}{\sin\pi x}=\frac{1}{1-\langle x\rangle}. \tag{3.34}$$

Clearly, the above result can be read as well from right to left, i.e. removal of brackets is equivalent to unwrapping S^1 into R^1, i.e. to switching from a given space, e.g. S^1, to its universal covering space, e.g. $\mathbf{R}^1$. By looking at Eq. (1.15) for expression of $\Gamma(z)$ and comparing it with Eq. (1.14) we notice that all singularities of $\Gamma(z)$ are exactly the same as those for $F(x)$. Hence, the same unwrapping is applicable for this case as well. These observations lead us to the following set of prescriptions:

(a) use Eq. (3.15) in Eq. (3.16) in order to obtain the full Veneziano-like amplitude,
(b) remove brackets,
(c) perform analytic continuation to negative values of c_i's,
(d) identify $-\frac{c_i}{N}$ with $-\alpha(i)$ ($i = s, t$ or u).

After this, let, for instance, $\alpha(s) = a + bs$, where both a and b are some positive (or better, non-negative) constants. Then, the tachyonic pole: $\alpha(s) = 0, n = 0$ (e.g. see Eqs. (1.17)–(1.19)) is killed by the corresponding phase factor in Eq. (3.33). The mass spectrum is determined by: (a) the actual numerical values of the constants a and b, (b) by the phase factors and (c) by the values of parameter N (even or odd).

For instance, the condition, Eq. (3.11b), leads to the requirement that the particle with masses satisfying equation $\alpha(s) = 2l, l = 1, 2, \ldots$ cannot be observed since the emerging pole singularities are killed by zeroes coming from the phase factor. In the case of 4-particle amplitude the inequality, Eq. (3.31), should be used with $n = 1$ thus leading to the constraints: $N \geq 3$ and $1 \leq m \leq 3$. If we choose $m = 3$ we obtain similar requirement forbidding particles with masses coming from the equation $\alpha(s) = 3l, l = 1, 2, \ldots$ (e.g. see the Remark 3.3).

Such limitations are not too severe, however. Indeed, let us consider for a moment the existing bosonic string parameters associated with the Veneziano amplitude. For the open string the known convention is: $\alpha(s) = 1 + \frac{1}{2}s$. The tachyon state is determined therefore by the condition: $\alpha(s) = 0$ thus producing $s = -2 = M^2$. If now $1 + \frac{1}{2}s = l$, we obtain $s = 2(l - 1), l = 1, 3, 5, \ldots$ (for $m = 2$) or $l = 1, 2, 4, 5, \ldots$ (for $m = 3$). Clearly, the combined use of these results produce the mass spectrum for the open bosonic string (without tachyons). If we want the graviton to be present in the spectrum we have to adjust the values of constants a and b. For instance, it is known [2] that for the closed string the tachyon occurs at $s = -8 = M^2$. This result can be obtained if we choose either $\alpha(s) = 2 + \frac{1}{4}s$ or $\alpha(s) = 1 + \frac{1}{8}s$. To decide which of these two expressions provides better fit to the experimental data we recall that the massless graviton should have spin equal to 2. If we want the graviton to be present in the spectrum we must select the first option. This is so because of the following arguments. First, we have to take into account that for large s and fixed t the amplitude $V(s, t)$ can be approximated by [2, p. 10],

$$V(s, t) \simeq \Gamma(-\alpha(t))(-\alpha(s))^{\alpha(t)}, \tag{3.35}$$

while the Regge theory predicts [2, pp. 3 and 4]that

$$V_J(s, t) = -\frac{g^2(-s)^J}{t - M_J^2} \simeq \frac{-g^2(-\alpha(s))^J}{\alpha(t) - J} \tag{3.36}$$

for the particle with spin J. This leaves us with the first option. Second, by selecting this option our task is not complete, since thus far we have ignored the actual value of the Fermat parameter N. Such ignorance causes emergence of the fictitious tachyon coming from the equation $2 + \frac{1}{4}s = l$ for $l = 1$. This difficulty is easily removable if we take into account that the “Shapiro–Virasoro” condition, Eq. (3.11b), is reducible to the “Veneziano condition, Eq. (3.11a), when all $\langle c_i \rangle$ in Eq. (3.11b) are even. At the same time, if N in Eq. (3.11a) is even, it can be brought to the form of Eq. (3.11b). Hence, in making identification of $-\frac{c_i}{N}$ with $-\alpha(i)$ we have to consider two options: (a) N is odd, then $\frac{c_i}{N} = \alpha(i)$ and (b) N is even, $N = 2\hat{N}$, then $\frac{c_i}{\hat{N}} = \alpha(i)$. Then, proceeding by analogy with arguments for the open bosonic string spectrum we reobtain the spectrum of the closed bosonic string”.

The fictitious tachyon is removed from the spectrum if we choose the option (b). Clearly, after this, in complete analogy with the “open string” case, we reobtain the tachyon-free spectrum of the “closed” bosonic string.

To complete our investigation of the Veneziano-like amplitudes we still would like to have some discussion related to Eqs. (3.35) and (3.36)). To this purpose, using integral representation of Γ given by

$$\Gamma(x) = \int_0^{\infty} \frac{\mathrm{d}t}{t} t^x \exp(-t)$$

and assuming that x is large and positive, the leading term of the saddle point approximation (to Γ) is readily obtained, and is given by

$$\Gamma(x) = Ax^x \exp(-x),$$

where A is some constant. Applying (with some caution) this result to

$$V(s, t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s) - \alpha(t))} \equiv B(-\alpha(s), -\alpha(t)),$$

we obtain Eq. (3.35).

Although such arguments formally explain the origin of the Regge asymptotic law, Eq. (3.35), they do not illuminate the combinatorial origin of this result essential for its generalization. To correct this deficiency we would like to use again Eq. (1.20). In the case when $z_0 = \cdots = z_n = 1$ the inner sum in the right-hand side yields the total number of monomials of the type $z_0^{k_0} \cdots z_n^{k_n}$ with $k_0 + \cdots + k_k = n$. The total number of such monomials is given by Eq. (1.21) which allows us to write the generating function $P(k, t)$, Eq. (1.22), and accordingly, the Veneziano amplitude, Eq. (1.23). The function $p(k, n)$ defined in Eqs. (1.21) and (1.25) happen to be the non-negative integer. In Section 1 we have mentioned already that it arises naturally as the dimension of the quantum Hilbert space associated (through the coadjoint orbit method) with the symplectic manifold of dimension $2k$ constructed by “inflating” Δ_k. We would like to use these observations now to complete our discussion of the Regge-like result, Eq. (3.35). To this purpose using Eq. (1.14) and assuming that $\alpha(t) \to k^*$ we can approximate the amplitude $V(s, t)$ by

$$V(s,t) \simeq -\frac{p_{\alpha(s)}(k^*)}{\alpha(t)-k^*} \simeq -\frac{p_{\alpha(s)}(\alpha(t))}{\alpha(t)-k^*}. \tag{3.37}$$

For large $\alpha(s)$[7] by combining Eqs. (1.25) and (3.37) we obtain

$$V(s,t) \simeq \frac{-1}{\alpha(t)-k^*}\frac{\alpha(s)^{k^*}}{k^*!}. \tag{3.38}$$

In view of the footnote remark, and taking into account that $k^* \simeq \alpha(t)$, this result coincides with Eq. (3.35) as required. In addition, however, for large k's it can be further rewritten as

$$V(s,t) \simeq \frac{-1}{\alpha(t)-k^*}\left(\frac{\alpha(s)}{k^*}\right)^{k^*} \simeq \frac{-1}{\alpha(t)-k^*}\left(\frac{\alpha(s)}{\alpha(t)}\right)^{\alpha(t)} \tag{3.39}$$

in accordance with Eq. (3.36). Obtained result is manifestly symmetric with respect to exchange $s \rightleftharpoons t$ in accordance with the earlier mentioned requirement $V(s, t) = V(t, s)$. Moreover, it explicitly demonstrates that the angular momentum of the graviton is indeed equal to 2 as required.

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